Integral Solutions - Integral One
Option Valuation Integrals

Gary Schurman MBE, CFA

We will define the variable θ to be a normally-distributed random variable with mean m and variance v. This statement in equation form is...

$$\theta \sim N\bigg[m, v\bigg] \tag{1}$$

We want to solve the following integrals...

$$I_1 = \int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\} A_0\,\mathrm{Exp}\left\{-\alpha\,t\right\}\delta\theta \tag{2}$$

$$I_2 = \int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\} A_0\,\mathrm{Exp}\left\{\theta\right\}\mathrm{Exp}\left\{-\alpha\,t\right\}\delta\theta \tag{3}$$

$$I_3 = \int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\} \left[A_0\,\mathrm{Exp}\left\{\theta\right\}\right]^2\delta\theta \tag{4}$$

We will define the function CNDF(x,n,v) to be the cumulative normal distribution function where the variable x is a normally-distributed random variable with mean m and variance v. This function tells us the probability that the random return θ_i drawn from a normal distribution with mean m and variance v will be less than some threshold value x. The equation for this function is...

$$\mathrm{CNDF}(x, m, v) = \mathrm{Prob}\bigg[\theta_i < x\bigg] = \int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\}\delta\theta \text{ ...where... } y \sim N\bigg[m, v\bigg] \tag{5}$$

The function in Excel that is the equivalent of Equation (5) above is...

$$\text{Excel function CNDF}(x, m, v) = \mathrm{NORMDIST}(x, m, \sqrt{v}, true) \tag{6}$$

The Solution To The First Integral

Note that we can rewrite integral Equation (2) above as...

$$I_1 = A_0\,\mathrm{Exp}\left\{-\alpha\,t\right\}\int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\}\delta\theta \tag{7}$$

Using Equation (5) above the solution to Equation (7) above is...

$$I_1 = A_0\,\mathrm{Exp}\left\{-\alpha\,t\right\}\mathrm{CNDF}(x, m, v) \tag{8}$$

The Solution To The Second Integral

Note that we can rewrite integral Equation (3) above as...

$$I_2 = A_0\,\mathrm{Exp}\left\{-\alpha\,t\right\}\int_{-\infty}^{x} \sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta^2 - 2\,m\,\theta + m^2)\right\}\mathrm{Exp}\left\{\theta\right\}\delta\theta \tag{9}$$

We can rewrite integral Equation (9) above as...

$$I_2 = A_0\,\mathrm{Exp}\left\{-\alpha\,t\right\}\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta^2-2\,m\,\theta+m^2-2\,v\,\theta)\right\}\delta\theta \tag{10}$$

We will define the following equation...

$$(\theta-m-v)^2=\theta^2-2\,v\,\theta-2\,m\,\theta+m^2+2\,m\,v+v^2 \tag{11}$$

Using Equation (11) above we can rewrite Equation (10) above as...

$$\begin{aligned}I_2 &= A_0\,\mathrm{Exp}\left\{-\alpha\,t\right\}\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,((\theta-m-v)^2-2\,m\,v-v^2)\right\}\delta\theta\\ &= A_0\,\mathrm{Exp}\left\{-\alpha\,t\right\}\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-m-v)^2+m+\frac{1}{2}\,v\right\}\delta\theta\\ &= A_0\,\mathrm{Exp}\left\{m+\frac{1}{2}\,v-\alpha\,t\right\}\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-(m+v))^2\right\}\delta\theta\end{aligned} \tag{12}$$

Using Equation (5) above the solution to Equation (12) above is...

$$I_2 = A_0\,\mathrm{Exp}\left\{m+\frac{1}{2}\,v-\alpha\,t\right\}\mathrm{CNDF}(x,m+v,v) \tag{13}$$

The Solution To The Third Integral

Note that we can rewrite integral Equation (4) above as...

$$I_3 = A_0^2\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta^2-2\,m\,\theta+m^2)\right\}\mathrm{Exp}\left\{2\,\theta\right\}\delta\theta \tag{14}$$

We can rewrite integral Equation (14) above as...

$$I_3 = A_0\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta^2-2\,m\,\theta+m^2-4\,v\,\theta)\right\}\delta\theta \tag{15}$$

We will define the following equation...

$$(\theta-m-2v)^2=\theta^2-4\,v\,\theta-2\,m\,\theta+m^2+4\,m\,v+4\,v^2 \tag{16}$$

Using Equation (16) above we can rewrite Equation (15) above as...

$$\begin{aligned}I_3 &= A_0^2\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,((\theta-m-2\,v)^2-4\,m\,v-4\,v^2)\right\}\delta\theta\\ &= A_0^2\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-m-2\,v)^2+2\,(m+v)\right\}\delta\theta\\ &= A_0^2\,\mathrm{Exp}\left\{2\,(m+v)\right\}\int\limits_{-\infty}^{x}\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\,(\theta-(m+2\,v))^2\right\}\delta\theta\end{aligned} \tag{17}$$

Using Equation (5) above the solution to Equation (17) above is...

$$I_3 = A_0^2\,\mathrm{Exp}\left\{2\,(m+v)\right\}\mathrm{CNDF}(x,m+2\,v,v) \tag{18}$$